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J. Differential Equations 190 (2003) 150–181
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Asymptotic behavior of solutions of quasilinear parabolic equations with supercritical nonlinearity☆

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Received December 5, 2001; revised December 5, 2001

Abstract

The Cauchy problem

$$\begin{cases} u_t - \Delta u^m = u^p & \text{in } \mathbf{R}^N \times (0, T), \\ u(x,0) = \tau u_0(x) & \text{in } \mathbf{R}^N, \end{cases} \tag{P}$$

is considered, where $\tau > 0$, $p > m \geqslant 1$ and $u_0(x)$ (> 0) is a bounded continuous radially symmetric function in $\mathbf{R}^N$. We choose p in some open interval (p_s, p_p) with $p_s = m(N+2)/[N-2]_+$ such that a peaking solution (incomplete blow-up solution) of (P) exists. Denote the solution of (P) by u_τ. We show that if $u_0(x)$ is nonincreasing in large $r = |x|$ and decays slowly: $u_0(x) = O(|x|^{-\alpha})$ as $|x| \to \infty$ $(2/(p-m) < \alpha)$, then u_τ is classified into one of the next three types according to the value τ as follows: There exists $\tau_1 \in (0, \infty)$ such that (I) u_τ blows up completely in finite time if $\tau > \tau_1$, (II) u_τ blows up incompletely in finite time and $||u_\tau(t)||_{L^\infty(\mathbf{R}^N)} = O(t^{-\frac{1}{p-1}})$ as $t \to \infty$ if $\tau = \tau_1$, (III) u_τ does not blow up in finite time and $||u_\tau(t)||_{L^\infty(\mathbf{R}^N)} = O(t^{-\frac{1}{p-1}})$ as $t \to \infty$ if $0 < \tau < \tau_1$.

Keywords: Blow-up; Parabolic equation; Supercritical nonlinearity; Asymptotic behavior; Decay slowly; Intersection number

☆Partially supported by Grant-in-Aid for Scientific Research (C) (No. 11640182) of Japan Society for the Promotion of Science.
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PII: S0022-0396(02)00086-4

1. Introduction

In this paper we shall consider the Cauchy problem

$$u_t - \Delta u^m = u^p \quad (x,t) \in \mathbf{R^N} \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R^N}, \tag{1.2}$$

where $u_t = \partial u/\partial t$, $m \geqslant 1$, $p > 1$, and $u_0(x) \geqslant 0$, $\in BC(\mathbf{R^N})$ (bounded continuous functions in $\mathbf{R^N}$). We shall only consider nonnegative solutions u. We are interested in the asymptotic behavior of the solutions.

It is well known that a unique nonnegative weak solution of (1.1) and (1.2) exists locally in time and can be extended as the time variable t increases as far as $u(\cdot,t) \in L^\infty(\mathbf{R^N})$ [3–6,19,23]. If we denote the solution of (1.1) and (1.2) by $u(x,t;u_0)$ and put

$$t_b(u_0) = \sup\{T \in (0,\infty);\; u(t;u_0) \in L^\infty((0,T);L^\infty)\} \tag{1.3}$$

(which is called the blow-up time of $u(x,t;u_0)$), then the following results hold [8–10,13,17,18,22]:

(I) Let $1 < p \leqslant m + 2/N$. Then all nontrivial solutions of (1.1) and (1.2) blow up in finite time. Namely, $t_b(u_0) < \infty$.
(II) Let $p > m + 2/N$. Then there exists a global solution of (1.1) and (1.2) when the initial data u_0 is sufficiently small. Namely, $t_b(u_0) = \infty$.

That is, $m + 2/N$ is the cutoff number between the blow-up case (I) and the global existence case (II).

Furthermore, when the initial data u_0 is radially symmetric, Galaktionov and Vazquez [11,12] study the blow-up phenomena more precisely as follows:

(A) Let $1 < p \leqslant p_s = m(N+2)/[N-2]_+$ where $[a]_+ = \max\{a,0\}$. Then, the blow-up solution $u(t;u_0)$ blows up completely at the blow-up time $t_b(u_0)$ $(<\infty)$:

$$u(x,t;u_0) = \infty \quad \text{in } \mathbf{R^N} \times (t_b(u_0),\infty). \tag{1.4}$$

Namely, if we put

$$t_c(u_0) = \inf\{T \in [0,\infty] \mid u(x,t;u_0) = \infty \text{ in } \mathbf{R^N} \times (T,\infty)\} \tag{1.5}$$

(which is called the complete blow-up time), then

$$t_c(u_0) = t_b(u_0).$$

Here we define the post-blow-up solution $u(x,t;u_0)$ in $\mathbf{R^N} \times (0,\infty)$ by the supremum of bounded subsolutions $v(x,t)$ of (1.1) and (1.2) satisfying $v(x,0) \leqslant u_0(x)$, at each point (x,t) (see [12,26]).

(B) Let $p_s < p < p_p$ where p_p is some constant defined in Section 4. Then, there exists an incomplete blow-up solution (called a peaking solution) which becomes finite after the blow-up time. That is, $t_b(u_0) < \infty$ and $t_c(u_0) = \infty$.

Namely, p_s is the cut off number between the complete blow-up case and the incomplete blow-up case when $p < p_p$.

Of course, in the supercritical case $p > p_s$, complete blow-up solutions also exist and some sufficient conditions on initial data for the complete blow-up are given by Galaktionov and Vazquez [12]. But, we do not know whether or not an incomplete blow-up solution exists when $p \geqslant p_p$.

Here, we mention the peaking solution $w_T(r,t)$ $(r = |x|)$ precisely. It is radially symmetric in x, blows up at the origin at $t = T$, decays to zero as $t \to \infty$ and is made of a backward self-similar blow-up solution with blow-up time T and a forward self-similar solution decaying to zero. Further, it satisfies

$$||w_T(\cdot,t)||_{L^\infty(\mathbf{R}^\mathbf{N})} \sim t^{\frac{-1}{p-1}} \quad \text{as } t \to \infty, \tag{1.6}$$

$$w_T(r,t) \sim r^{-\frac{2}{p-m}} \quad \text{as } r \to \infty \ \text{ for } t \geqslant 0. \tag{1.7}$$

We note that the exponent $\frac{2}{p-m}$ is critical in the next sense:

(i) If

$$\liminf_{r \to \infty} r^{\frac{2}{p-m}} u_0(r) > C \tag{1.8}$$

for some $C > 0$, then $t_b(u_0) < \infty$ (see [20,25]).

(ii) If $u_0(x) \leqslant w_T(x,T) = c_1 |x|^{-\frac{2}{p-m}}$ $(c_1 > 0)$ and $u_0(x) \not\equiv w_T(x,T)$ then $t_b(u_0) = \infty$ (see [12]).

So, throughout this paper, in order to ensure the existence of a global solution of (1.1) and (1.2) with initial data τu_0 for small $\tau > 0$, we assume the following condition on u_0: Let $\alpha \in (\frac{2}{p-m}, N)$. There exists a constant $C > 0$ such that

$$u_0(x)|x|^\alpha \leqslant C \quad \text{for } x \in \mathbf{R}^\mathbf{N}. \tag{1.9}$$

Further, throughout this paper we use the following notations. For two functions $f(t)$ and $g(t)$, we say that $f(t) = O(g(t))$ as $t \to \infty$ if $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$ and that $f(t) = o(g(t))$ as $t \to \infty$ if $\limsup_{t\to\infty} |f(t)/g(t)| = 0$. Further, we say that $f(t) \sim g(t)$ as $t \to \infty$ if $f(t) = O(g(t))$ as $t \to \infty$ and $g(t) = O(f(t))$ as $t \to \infty$. $L^q (1 \leqslant q \leqslant \infty)$ is the usual space of all L^q-functions in $\mathbf{R}^\mathbf{N}$ with norm $||f||_q \equiv ||f||_{L^q(\mathbf{R}^\mathbf{N})}$.

Our aim of this paper is to study the asymptotic behavior solutions of (1.1) and (1.2) more precisely in the supercritical case when the initial data $u_0(x) = u_0(r)$ $(r = |x|)$ is a radially symmetric function and satisfies condition (1.9). Assume that for some $r_0 > 0$,

$$u_0(r) \quad \text{is nonincreasing in } r \geqslant r_0 \tag{1.10}$$

and when $m > 1$ we further assume

$$u_0(r) > 0 \quad \text{in } [0, r_0]. \tag{1.11}$$

Then, when $p_s < p < p_p$ we show that $u(t;\tau u_0)$ $(u_0(x) \not\equiv 0)$ is classified into one of the next three types according to the value of $\tau > 0$ as follows: There exists $\tau_1 \in (0,\infty)$ such that

(I) $t_c(\tau u_0) < \infty$, i.e. $u(t;\tau u_0)$ blows up completely in finite time if $\tau > \tau_1$,
(II) $t_b(\tau u_0) < \infty$, $t_c(\tau u_0) = \infty$ and $||u(t;\tau u_0)||_\infty = O(t^{-\frac{1}{p-1}})$ if $\tau = \tau_1$,
(III) $t_b(\tau u_0) = \infty$ and $||u(t;\tau u_0)||_\infty = O(t^{-\frac{1}{p-1}})$ if $0 < \tau < \tau_1$.

When $m = 1$ and the initial data has the compact support, similar results were obtained by Mizoguchi [21]. But, in Types II and III the order of the decay rate of $||u(t;\tau u_0)||_\infty$ as $t \to \infty$ was not obtained there. Her methods are based mainly on comparing solutions with radially symmetric stationary solutions of (1.1) and using the energy methods. But, in our proof we only compare solutions with some incomplete blow-up solution with initial data $u_0(x) = \min\{h, k|x|^{-\alpha}\}$, which is like a peaking solution. It seems that her methods cannot be applied directly to our case where the initial data decay slowly. But, our methods of the proof are essentially based on the idea of the proof of Theorem 15.1 of [12], as in the proof of Lemma 4.1 of [21].

Thus, our methods are based on the comparison theorem and comparing solutions with some incomplete blow-up solution with initial data $u_0(x) = \min\{h, k|x|^{-\alpha}\}$, which is constructed by the methods using a peaking solution $w_T(r,t)$. Then, as in [12,21], we make use of the nonincrease of intersection number between a solution of (1.1) and the incomplete blow-up solution (or a peaking solution $w_T(r,t)$). Hence, we must restrict ourselves to radially symmetric solutions which is nonincreasing in large $r = |x|$.

In the subcritical case $m < p < p_s$, there are some papers studying these problems. Especially, when $m = 1$, Kawanago [16] obtained the following very interesting results: He clarifies the structure of the space of positive solutions of (1.1) with the initial data $u_0(x)(\not\equiv 0)$ decaying exponentially as $|x| \to \infty$. Namely, $u(t;\tau u_0)$ is classified into one of the next three types according to the value of $\tau > 0$ as follows: There exists $\tau_1 > 0$ such that

(I) $t_b(\tau u_0) < \infty$, i.e. $u(t;\tau u_0)$ blows up in finite time if $\tau > \tau_1$,
(II) $t_b(\tau u_0) = \infty$ and $||u(t;\tau u_0)||_\infty \sim t^{-1/(p-1)}$ as $t \to \infty$ if $\tau = \tau_1$,
(III) $t_b(\tau u_0) = \infty$ and $||u(t;\tau u_0)||_\infty \sim t^{-N/2}$ as $t \to \infty$ if $0 < \tau < \tau_1$.

We note that in these results the radial symmetry of solutions is not assumed.

When $m \geqslant 1$ and the initial data u_0 decays more slowly, Suzuki [25] also studies these problems. He extends the Kawanago's results partially.

The rest of the paper is organized as follows. In Section 2, we define a weak solution of (1.1) and state the main results (Theorem 2.5 and Corollary 2.6). In Section 3, we summarize preliminary lemmas and propositions, and in Section 4 we introduce the peaking solution $w_T(x,t)$, which is made in [12]. The existence of an incomplete blow-up solution with initial data $u_0(x) = \min\{h, k|x|^{-\alpha}\}$ is shown in Section 5. Finally, in Section 6 we prove Theorem 2.5 and Corollary 2.6.

2. Definitions and main results

In this section, we state the definition of a weak solution and a post-blow-up solution of (1.1) (see [26]), and state the main results. Let Ω be a domain in $\mathbf{R}^{\mathbf{N}}$.

Definition 2.1. A function $u \in L^\infty(\Omega \times (0,\tau))$ for each $0<\tau<T$ is called a *weak solution of* (1.1) *in* $\Omega \times (0,T)$, if it satisfies

(i) $u(x,t) \geqslant 0$ in $\Omega \times [0,T)$ and $\in B(\bar{\Omega} \times (0,\tau))$ (bounded continuous) for each $0<\tau<T$,

(ii)
$$u(t) \to u(0) \quad \text{in } L^1_{\mathrm{loc}}(\bar{\Omega}) \text{ as } t \downarrow 0, \tag{2.1}$$

(iii) For any bounded domain $D \subset \Omega$ with smooth boundary ∂D, $0<\tau<T$ and nonnegative $\phi(x,t) \in C^2(\bar{D} \times [0,T))$ which vanishes on the boundary ∂D,
$$\begin{aligned}\int_D u(x,\tau)\phi(x,\tau)\,dx - \int_D u(x,0)\phi(x,t)\,dx \\ = \int_0^\tau \int_D \{u\phi_t + u^m \Delta\phi + u^p\phi\}\,dx\,dt - \int_0^\tau \int_{\partial D} u \partial_n \phi\,dS\,dt,\end{aligned} \tag{2.2}$$
where n denotes the outer unit normal to the boundary.

A *supersolution* (or *subsolution*) of (1.1) in $\Omega \times (0,T)$ is defined by (i)–(iii) of Definition 2.1 with equality (2.1) replaced by $\geqslant$ (or $\leqslant$).

As mentioned in Section 1, if $u_0 \in L^\infty(\mathbf{R}^{\mathbf{N}})$ then problem (1.1), (1.2) has a unique local weak solution $u(x,t)$ in time, and if $u_0 \in C(\mathbf{R}^{\mathbf{N}})$ then $u(x,t) \in C(\mathbf{R}^{\mathbf{N}} \times [0,T))$. A post-blow-up solution is defined as follows.

Definition 2.2. (I) A function $u(x,t)\colon\ \mathbf{R}^{\mathbf{N}} \times (0,\infty) \to [0,\infty]$ is called a *semi-solution* of (1.1) and (1.2) in $\mathbf{R}^{\mathbf{N}} \times (0,\infty)$ if it satisfies

(i) for any bounded domain D with smooth boundary and closed interval $[t_0, t_1]$ satisfying $\bar{D} \times [t_0, t_1] \subset \{\mathbf{R}^\mathbf{N} \times [0, \infty)\} \backslash \tilde{B}$, u is a weak solution of (1.1) in $D \times (t_0, t_1)$ where

$$\tilde{B} = \{(x_0, t_0) \in \mathbf{R}^\mathbf{N} \times [0, \infty) \mid \text{there is } (x_n, t_n) \in \mathbf{R}^\mathbf{N} \times [0, \infty) \text{ such that } x_n \to x_0, t_n \to t_0, u(x_n, t_n) \to \infty \text{ as } n \to \infty\}. \tag{2.3}$$

(ii) For any bounded subsolution $v(x,t)$ of (1.1) in $\mathbf{R}^\mathbf{N} \times (t_0, t_1)$ $(0 \leqslant t_0 < t_1 < \infty)$ and $v(x, t_0) \leqslant u(x, t_0)$ in $x \in \mathbf{R}^\mathbf{N}$, $v(x,t) \leqslant u(x,t)$ in $\mathbf{R}^\mathbf{N} \times (t_0, t_1)$.

(II) The minimal semi-solution $u(x,t)$ of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, \infty)$ is called a *weak solution of* (1.1) and (1.2) *in* $\mathbf{R}^\mathbf{N} \times (0, \infty)$. That is, weak solution u is a semi-solution of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, \infty)$ and for any semi-solution v of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, \infty)$, $u(x,t) \leqslant v(x,t)$ in $\mathbf{R}^\mathbf{N} \times (0, \infty)$.

Remark 2.3. Let $u(x,t)$ be a weak solution of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, \infty)$ which is defined by Definition 2.2. Then, u is a weak solution of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, T)$ in the sense of Definition 2.1 when $u \in L^\infty(\mathbf{R}^\mathbf{N} \times (0, T))$.

Let $u(x,t)$ be a weak solution of (1.1) and (1.2) in $\mathbf{R}^\mathbf{N} \times (0, \infty)$. Let $t_b(u_0)$ and $t_c(u_0)$ be defined by (1.1) and (1.2), respectively. It is well known that if $u_0 \in L^\infty(\mathbf{R}^\mathbf{N})$ then $t_b > 0$. When $t_b < \infty$ we say that *u blows up* at *the blow-up time* t_b. Then we have by the existence and uniqueness theorem,

$$\lim_{t \uparrow t_b} ||u(t)||_{L^\infty(\mathbf{R}^\mathbf{N})} = \infty. \tag{2.4}$$

When $t_c < \infty$ we say that *u blows up completely* at *the complete blow-up time* t_c. Clearly, $t_b \leqslant t_c$.

We can obtain the existence, uniqueness and some properties of a post-blow-up solution ([26]).

Proposition 2.4. *Let* $0 \leqslant u_0(x) \leqslant \infty$ *in* $\mathbf{R}^\mathbf{N}$. *Then, there exists a unique weak solution* $u(x,t) = u(x,t;u_0)$ *of* (1.1) *and* (1.2) *in* $\mathbf{R}^\mathbf{N} \times (0, \infty)$. *Furthermore, it has the following properties*:

(i) *If we put* $S(t)u_0 = u(t; u_0)$, *then*

$$S(t)S(\tau) = S(t+\tau) \quad \textit{for } t, \tau \geqslant 0 \quad \textit{and} \quad S(0) = I(\textit{identity map}). \tag{2.5}$$

(ii) *For any bounded subsolution* $v(x,t)$ *of* (1.1) *in* $D \times (t_0, t_1)$ *where* $D = \mathbf{R}^\mathbf{N}$ *or* D *is a bounded domain in* $\mathbf{R}^\mathbf{N}$ *which satisfies* $v(x,t) \leqslant u(x,t)$ *on the parabolic boundary of* $D \times (t_0, t_1)$, *we have* $v(x,t) \leqslant u(x,t)$ *in the whole* $D \times (t_0, t_1)$.

(iii)

$$\mathbf{L}^{\mathbf{N}}(B(t)) = 0 \quad \textit{for } 0 < t < t_c, \tag{2.6}$$

where $\mathbf{L}^{\mathbf{N}}(K)$ *is the Lebesgue measure of set* $K \subset \mathbf{R}^{\mathbf{N}}$ *and*

$$B(t) = \{x \in \mathbf{R}^{\mathbf{N}}; u(x,t) = \infty\}.$$

Further, we have for any bounded star-shaped domain $D \subset \Omega$ *with smooth boundary,*

$$\sup_{0<t<T} \int_D \psi_D(x) u(x,t)\, dx < \infty \quad \text{for } T < t_c, \tag{2.7}$$

where ψ_D *is the first eigenfunction of* $-\Delta$ *in D with Dirichlet boundary condition* (ψ_D *is normalized:* $\int_D \psi_D\, dx = 1$). *Moreover, if* $t_c = \infty$ *then there is* $\mu_D > 0$ *independent of u such that*

$$\int_D \psi_D(x) u(x,t)\, dx \leqslant \mu_D \quad \textit{for } t > 0. \tag{2.8}$$

Proof. See [26]. □

Now, we shall state the main results of this paper. For this aim, we introduce several spaces of functions as follows. When $\alpha \in (0, N)$, let $L^\infty_\alpha = \{f \in L^\infty; \|f\|_{\infty,\alpha} \equiv \sup_{x \in \mathbf{R}^{\mathbf{N}}} (|x|+1)^\alpha |f| < \infty\}$, which is a Banach space with norm $\|\cdot\|_{\infty,\alpha}$. Let $\mathbf{S} = \{f \in L^\infty; f(x)$ is a radially symmetric function in $\mathbf{R}^{\mathbf{N}}\}$,

$$\begin{aligned} X_\alpha = \{ & f \in L^\infty_\alpha \cap C(\mathbf{R}^{\mathbf{N}}) \cap \mathbf{S} \,|\, f(r) = f(x) \geqslant 0 \text{ in } r = |x| \geqslant 0 \text{ and there} \\ & \text{exists } r_0 > 0 \text{ such that for each } r' \geqslant r_0, \sigma_{r'} f(r) \geqslant f(r) \text{ in } r \geqslant r'\}, \end{aligned} \tag{2.9}$$

where $\sigma_{r'} f(r)$ is the reflection of $f(r)$ with respect to r', namely, $\sigma_{r'} f(r) = f(2r' - r)$ and

$$\begin{aligned} X^+_\alpha = \{ & f \in X_\alpha; \text{ for some } r_0 > 0, f(r) \text{ is} \\ & \text{nonincreasing in } r \geqslant r_0 \text{ and } f(r) > 0 \text{ in } [0, r_0]\}. \end{aligned} \tag{2.10}$$

Clearly, $X^+_\alpha \subset X_\alpha$, and X_α and X^+_α are cones of the Banach space $L^\infty_\alpha \cap C(\mathbf{R}^{\mathbf{N}})$ with norm $\|\cdot\| := \|\cdot\|_{\infty,\alpha}$.

We set

$$K_\alpha = \{u_0 \in X_\alpha, \ ; t_c(u_0) = \infty\}, \tag{2.11}$$

$$C_\alpha = X_\alpha \backslash K_\alpha = \{u_0 \in X_\alpha \ ; t_c(u_0) < \infty\}. \tag{2.12}$$

We denote by ∂K_α the boundary of K_α in X_α and $\mathrm{Int}(K_\alpha)$ the interior of K_α in X_α. Put

$$p_s = m \times \frac{N+2}{[N-2]_+} \quad (\text{when } 1 \leqslant N \leqslant 2,\ p_s = \infty), \tag{2.13}$$

$$p_p = 1 + \frac{3m + [(m-1)^2(N-10)^2 + 2(m-1)(5m-4)(N-10) + 9m^2]^{1/2}}{[N-10]_+} \tag{2.14}$$

(when $1 \leqslant N \leqslant 10$, $p_p = \infty$), where $[a]_+ = \max\{a, 0\}$ and p_p is introduced by Galaktionov and Vazquez [12].

Theorem 2.5. *Suppose $p_s < p < p_p$ and $2/(p-m) < \alpha < N$. Then we obtain the following*:

(i) *K_α is an unbounded closed subset in X_α and $0 \in \mathrm{Int}(K_\alpha)$. $\mathrm{Int}(K_\alpha)$ and C_α are unbounded open subsets in X_α.*
(ii) *If $u_0 \in X_\alpha$ then $u(t; u_0) \in X_\alpha$ for each $t \in (0, t_b(u_0))$.*
(iii) *For any $u_0 \in X_\alpha^+$, there exists unique $\tau_0 \in (0, \infty)$ such that*

$$\tau u_0 \in \begin{cases} \mathrm{Int}(K_\alpha) & \text{if } \tau \in (0, \tau_0), \\ \partial K_\alpha & \text{if } \tau = \tau_0, \\ C_\alpha & \text{if } \tau \in (\tau_0, \infty). \end{cases} \tag{2.15}$$

Furthermore, $G_\alpha^+ = \{u_0 \in X_\alpha^+; ||u_0|| = 1\}$ and $\partial K_\alpha \cap X_\alpha^+$ are homeomorphic by $P|_{G_\alpha^+}$ where $P : G_\alpha^+ \to \partial K_\alpha \cap X_\alpha^+$ is the well-defined projection: $Pu_0 = \tau_0 u_0 \in \partial K_\alpha \cap X_\alpha^+$.

(iv) *If $u_0 \in K_\alpha$ then*

$$||u(t; u_0)||_\infty = O(t^{-1/(p-1)}) \quad \text{as } t \to \infty. \tag{2.16}$$

(v) *If $u_0 \in \mathrm{Int}(K_\alpha)$ then $t_b(u_0) = \infty$ and if $u_0 \in \partial K_\alpha$ then $t_b(u_0) < \infty$.*

Corollary 2.6. *When $m = 1$, in Theorem 2.5 we can replace X_α and X_α^+ by $\tilde{X}_\alpha$ and $\tilde{X}_\alpha \backslash \{0\}$, respectively, where*

$$\tilde{X}_\alpha = \{f \in L_\alpha^\infty \cap C(\mathbf{R^N}) \cap \mathbf{S};\ f \geqslant 0 \text{ in } \mathbf{R^N} \text{ and } f(x) = f(r)(r = |x|)$$
$$\text{is nonincreasing in } r \geqslant r_0 \text{ for some } r_0 > 0\}. \tag{2.17}$$

Remark 2.7. As mentioned in Section 1, in the subcritical case $m < p < p_s$ similar results were obtained by Kawanago [16] (when $m = 1$) and Suzuki [25] (when $m \geqslant 1$).

In the rest of this section, we state the fundamental properties of the post-blow-up solution, which are proved by Suzuki [26].

Proposition 2.8. *Let* $0 \leqslant u_0(x), v_0(x) \leqslant \infty$. *Let* u *and* v *be weak solutions of* (1.1) *and* (1.2) *in* $\mathbf{R^N} \times (0, \infty)$ *with initial data* $u_0(x)$ *and* $v_0(x)$, *respectively. If* $u_0(x) \leqslant v_0(x)$ *in* $\mathbf{R^N}$, *then* $u(x,t) \leqslant v(x,t)$ *in the whole* $\mathbf{R^N} \times (0, T)$.

Proposition 2.9. *For* $u_0(x) \in [0, \infty]$, *let* $\{u_{0,n}\} \subset L^\infty(\mathbf{R^N})$ *satisfy that* $u_{0,n}(x) \geqslant 0$ *in* $\mathbf{R^N}$ *and* $u_{0,n}(x) \uparrow u_0(x)$ *as* $n \to \infty$ *for each* $x \in \mathbf{R^N}$. *Let* u *and* u_n *be weak solutions of* (1.1) *and* (1.2) *in* $\mathbf{R^N} \times (0, \infty)$ *with initial data* $u_0(x)$ *and* $u_{0,n}(x)$, *respectively. Then*

$$u_n(x,t) \uparrow u(x,t) \text{ as } n \to \infty \text{ for each } (x,t) \in \mathbf{R^N} \times (0, \infty). \tag{2.18}$$

3. Preliminary

In this section, in order to show the theorem we state the properties of solutions of (1.1) and (1.2) with the initial data $u_0(x) \in X_\alpha$.

Lemma 3.1. *Suppose* $0 < \alpha < N$ *and* $u_0 \in X_\alpha$. *Let* $u(x,t)$ *be a weak solution of* (1.1) *and* (1.2) *in* $\mathbf{R^N} \times (0, T)$. *Then*

$$u(\cdot, t) \in X_\alpha \quad \text{for } t \in (0, t_b(u_0)). \tag{3.1}$$

Proof. Let $u_0 \in X_\alpha$. Since $u_0(x)$ is a radially symmetric function in $x \in \mathbf{R^N}$, the uniqueness of solutions implies that for each $t \in (0, t_b(u_0))$ $u(x,t)$ is also a radially symmetric function in $x \in \mathbf{R^N}$. Further, since Eq. (1.1) is invariant under the reflection of $x \in \mathbf{R^N}$ in a hyperplain, the comparison principle implies that $u(\cdot, t) \in X_\alpha$ for $t \in (0, t_b(u_0))$. $\square$

Lemma 3.2. *Suppose* $0 < \alpha < N$ *and* $u_0 \in X_\alpha$. *Let* $u(x,t) = u(r,t)$ $(r = |x|)$ *be a weak solution of* (1.1) *and* (1.2) *in* $\mathbf{R^N} \times (0, \infty)$. *Let* $0 < T < t_c(u_0)$. *Then, there exist constants* $r_1 > 0$ *and* $C_1 > 0$ *such that*

$$u(r,t) \leqslant C_1 r^{-\alpha} \quad \text{in } [r_1, \infty) \times [0, T]. \tag{3.2}$$

Furthermore, if there exists constants $\alpha' \in [\alpha, N)$ *and* $c_0 > 0$ *such that*

$$u_0(r) \geqslant c_0 \langle r \rangle^{-\alpha'} \quad \text{in } [0, \infty) \times [0, T], \tag{3.3}$$

where $\langle r \rangle = \sqrt{1 + r^2}$, *then there exists* $c_1 > 0$ *such that*

$$u(r,t) \geqslant c_1 \langle r \rangle^{-\alpha'} \quad \text{in } [0, \infty) \times [0, T]. \tag{3.4}$$

Proof. Since $u_0 \in X_\alpha$, $u_0(x)$ satisfies condition (H) for a domain $\{|x| < r_0\}$, where (H) (see (H1) and (H2)) is as in [26, Section 2]. Hence, by Lemma 4.3 of [26] we see that for each $r_1 > r_0$,

$$u(x,t) \in L^\infty(\{r \geqslant r_1\} \times [0,T]). \tag{3.5}$$

Therefore, similarly as in the proof of Lemma 3.4 of [26], we can get (3.2).

Next, we assume (3.3) and show (3.4). Let $W_{\alpha'}(x,t)$ in $\mathbf{R}^\mathbf{N} \times (0,\infty)$ be the weak solution of the problem

$$\begin{cases} w_t - \Delta w^m = 0 & \text{in } (x,t) \in \mathbf{R}^\mathbf{N} \times (0,\infty), \\ w(x,0) = A|x|^{-\alpha'} & \text{in } x \in \mathbf{R}^\mathbf{N}, \end{cases} \tag{3.6}$$

where $A > 0$. Then, it is well known that $W_{\alpha'}(x,t)$ is represented by the next form (see [15,25]):

$$W_{\alpha'}(x,t) = A(A^{m-1}t)^{\frac{-\alpha'}{\alpha'(m-1)+2}} h(\eta) = A^{\frac{2}{\alpha'(m-1)+2}} t^{\frac{-\alpha'}{\alpha'(m-1)+2}} h(\eta), \tag{3.7}$$

where $\eta = |x|/(A^{m-1}t)^{\frac{1}{\alpha'(m-1)+2}}$ and $h(\eta) > 0$ $(\eta \geqslant 0)$ is a nonincreasing continuous function satisfying

$$\lim_{\eta\to\infty} \eta^{\alpha'} h(\eta) = 1. \tag{3.8}$$

Hence, for some $C > 0$,

$$h(\eta) \leqslant C\eta^{-\alpha'} \quad \text{for } \eta \geqslant 0, \tag{3.9}$$

from which,

$$W_{\alpha'}(r,1) \leqslant CAr^{-\alpha'} \quad \text{for } r > 0 \tag{3.10}$$

and if A is small enough then

$$W_{\alpha'}(r,1) \leqslant CAr^{-\alpha'} \leqslant c_0 \langle r \rangle^{-\alpha'} \quad \text{for } r \geqslant 1. \tag{3.11}$$

On the other hand, if A is small enough then

$$W_{\alpha'}(r,1) \leqslant A^{\frac{2}{\alpha'(m-1)+2}} h(0) \leqslant c_0 \langle r \rangle^{-\alpha'} \quad \text{for } 0 \leqslant r \leqslant 1. \tag{3.12}$$

Combining above inequalities we have for small $A > 0$,

$$W_{\alpha'}(r,1) \leqslant c_0 \langle r \rangle^{-\alpha} \quad \text{for } r \geqslant 0. \tag{3.13}$$

Therefore, since $W_{\alpha'}(r, 1+t)$ is a subsolution of (1.1) in $\{r \geqslant 0\} \times [0,\infty)$, we have by the comparison theorem (Proposition 2.4(ii)),

$$W_{\alpha'}(x, t+1) \leqslant u(x,t) \quad \text{in } (x,t) \in \mathbf{R}^\mathbf{N} \times (0,T]. \tag{3.14}$$

Noting (3.8) we see (3.4). $\square$

The next proposition shows the nonincreasing character of the intersection number between some two solutions of (1.1) in time whose initial data belong to X_α, and this character plays an important role in the proof of the theorem.

Proposition 3.3. *Let* $0<\alpha_3\leqslant\alpha_1<\alpha_2<N$. *Let* $u_0(x)=u_0(r), v_0(x)=v_0(r)\in X_{\alpha_3}$ $(r=|x|)$ *satisfy that* $v_0(r)>0$ *in* $r\geqslant 0$, *for some* $C>1$

$$u_0(r)\leqslant Cr^{-\alpha_2} \quad \text{in } r>0, \tag{3.15}$$

$$v_0(x)\geqslant C^{-1}r^{-\alpha_1} \quad \text{in } r\geqslant 1, \tag{3.16}$$

and for some $r_0\in(0,\infty]$

$$u_0(r)>0 \text{ in } r<r_0 \quad \text{and} \quad u_0(r)=0 \text{ in } r\geqslant r_0. \tag{3.17}$$

Assume $u_0(0)>v_0(0)$ *and assume that* $u_0(r)$ *and* $v_0(r)$ *intersect at only one point in* $r>0$. *Let* $u(r,t)$ *and* $v(r,t)$ *be bounded weak solutions of* (1.1) *in* $\mathbf{R}^{\mathbf{N}}\times(0,T)$ *with initial data* $u_0(r)$ *and* $v_0(r)$, *respectively. Then, if*

$$u(0,t_1)=v(0,t_1) \tag{3.18}$$

for some $t_1\in(0,T)$,

$$u(r,t)\leqslant v(r,t) \quad \text{in } r\geqslant 0,\ t\in[t_1,T). \tag{3.19}$$

Proof. We first consider the case $r_0=\infty$. Then, by (3.17) and the positivity of the solution (see [2]) we obtain

$$u(r,t)>0 \quad \text{in } [0,\infty)\times[0,T), \tag{3.20}$$

$$v(r,t)>0 \quad \text{in } [0,\infty)\times[0,T). \tag{3.21}$$

Further, by the assumptions, there exists $t_2\in(0,t_1]$ such that

$$u(0,t)>v(0,t) \text{ in } [0,t_2) \quad \text{and} \quad u(0,t_2)=v(0,t_2). \tag{3.22}$$

Put $w(r,t)=u(r,t)-v(r,t)$. Then, because of Lemma 3.2, we have for some $r_1>0$,

$$w(r,t)<0 \quad \text{in } [r_1,\infty)\times[0,T). \tag{3.23}$$

On the other hand, it follows from (3.22) that for any $T'\in(0,t_2)$ there exists $\delta\in(0,r_1)$ such that

$$w(r,t)>0 \quad \text{in } [0,\delta]\times[0,T']. \tag{3.24}$$

Hence, put

$$z(t) = \#\{r \in [\delta, r_1] \mid w(r,t) = 0\} \quad \text{for } t \geqslant 0. \tag{3.25}$$

Then, $z(0) = 1$ by the assumption and so $z(t) = 1$ in $[0, T']$ by Theorem B of Angenent [1], since $w(\delta, t) \neq 0$ and $w(r_1, t) \neq 0$ in $[0, T']$, and $w(r,t)$ is a solution of some parabolic equation in $[\delta, r_1] \times [0, T']$. Namely, putting

$$\tilde{z}(t) = \#\{r \in [0, \infty) \mid w(r,t) = 0\}, \tag{3.26}$$

we have $\tilde{z}(t) = 1$ in $[0, t_2)$. Hence, applying Theorem B of Angenent [1] again to $w(r,t)$, we find a C^1-function $r = g(t) \in (0, r_1)$ for $t \in (0, t_2)$ such that

$$\{r \in [0, \infty) \mid w(r,t) = 0\} = \{r \mid r = g(t)\} \quad \text{for each } t \in (0, t_2). \tag{3.27}$$

Put

$$r' = \liminf_{t \uparrow t_2} g(t). \tag{3.28}$$

Then, $r' \in [0, r_1]$ and by the continuity of u,

$$w(r, t_2) \leqslant 0 \quad \text{in } [r', \infty). \tag{3.29}$$

Therefore, we shall show

$$w(r, t_2) \leqslant 0 \quad \text{in } [0, r']. \tag{3.30}$$

If $r' = 0$, then (3.30) is obvious. So, let $r' > 0$. Assume that (3.30) does not hold. Then, there exists $r_2 \in (0, r')$ such that

$$w(r_2, t_2) > 0. \tag{3.31}$$

Hence, there exists $\delta > 0$ such that

$$w(r_2, t) > 0 \quad \text{for } |t - t_2| \leqslant \delta, \tag{3.32}$$

$$w(r, t_2 - \delta) > 0 \quad \text{in } [0, r_2]. \tag{3.33}$$

Since $w(x,t)$ is a solution of some parabolic equation in $\{|x| < r_2\} \times \{|t - t_2| < \delta\}$, the maximum principle implies

$$w(x,t) > 0 \quad \text{in } |x| < r_2, \ \ |t - t_2| < \delta. \tag{3.34}$$

This is a contradiction to $w(0, t_2) = u(0, t_2) - v(0, t_2) = 0$. So, we obtain (3.30). Therefore,

$$w(r, t_2) \leqslant 0 \quad \text{in } r \in [0, \infty), \tag{3.35}$$

that is,

$$u(r,t_2) \leqslant v(r,t_2) \quad \text{in } r \in [0,\infty). \tag{3.36}$$

Thus, by the comparison theorem we obtain

$$u(r,t) \leqslant v(r,t) \quad \text{in } r \geqslant 0, \ \ t \in (t_2,T). \tag{3.37}$$

Next, we consider the case $r_0 < \infty$. When $m = 1$, by the positivity of solutions we see that $u(x,t) > 0$ in $\mathbf{R}^\mathbf{N} \times (0,T)$. Hence, the case $m = 1$ is reduced to the case $r_0 = \infty$. So, we consider the case $m > 0$. Then we note that $\operatorname{supp} u(\cdot,t)$ (the support of $u(\cdot,t)$) spreads out continuously to $\mathbf{R}^\mathbf{N}$ as $t \to \infty$. Let t_2 satisfy (3.22). Then, there exist constants $r_1 \in (0,r_0)$ and $h \in (0,t_2)$ such that

$$u(r,t) > 0 \quad \text{in } [0,r_1] \times [0,h], \tag{3.38}$$

$$w(r,t) < 0 \quad \text{in } [r_1,\infty] \times [0,h]. \tag{3.39}$$

Therefore, similarly, as in the case $r_0 = \infty$, there exists C^1-function $r = g(t) \in (0,r_1)$ in $[0,h]$ such that

$$\{r \in [0,\infty) \mid w(r,t) = 0\} = \{r \mid r = g(t)\} \quad \text{for each } t \in [0,h]. \tag{3.40}$$

Repeating this operation in time, we can extend $g(t)$ to $0 < t < t_2$ to satisfy

$$\{r \in [0,\infty) \mid w(r,t) = 0\} = \{r \mid r = g(t)\} \quad \text{for each } t \in [0,t_2). \tag{3.41}$$

Thus, similarly, as in the case $r_0 = \infty$, we obtain (3.37). $\square$

Proposition 3.4. *Let $0 < \alpha < N$.*

(i) *Let $u_0 \in X_\alpha \backslash \{0\}$ and $t_c(u_0) = \infty$. Put $u_{0,\varepsilon} = [u_0 - \varepsilon]_+$ and $u_\varepsilon(x,t) = u(x,t;u_{0,\varepsilon})$ for $\varepsilon \geqslant 0$. Then, there exists constants $r_0 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$, $u_{0,\varepsilon} \not\equiv 0$ and*

$$u_\varepsilon(r,t) < \infty \quad \textit{in } (r_0,\infty) \times [0,\infty), \tag{3.42}$$

and for each $t \geqslant 0$

$$u_\varepsilon(r,t) \ \textit{is a nonincreasing function in } r \geqslant r_0, \tag{3.43}$$

and if $u_\varepsilon(r,t) > 0$ for some $r \in (r_0,\infty)$ and $t \in (0,\infty)$, then

$$\frac{\partial u_\varepsilon}{\partial r}(r,t) < 0. \tag{3.44}$$

Furthermore, there exist constants $t_0 > 0$ and $\delta > 0$ such that for $\varepsilon \in [0,\varepsilon_0]$,

$$u_\varepsilon(x,t_0) \geqslant \delta \quad \textit{in } |x| \leqslant r_0. \tag{3.45}$$

(ii) *When* $m = 1$, *the above arguments hold with* X_α *replaced by* $\tilde{X}_\alpha$. *Hence, Lemmas* 3.1 *and* 3.2 *hold with* X_α *replaced by* $\tilde{X}_\alpha$.

Proof. (i) Eq. (3.42) follows from Lemma 3.2, and by the similar methods to those of the proof of Lemmas 3.1 and 3.2, it is not difficult to see (3.43). Eq. (3.44) follows from the same methods as those of Friedman–Mcleod [7] (see also the proof of (ii) below). Eq. (3.45) is clear, since $\operatorname{supp} u_\varepsilon(\cdot, t)$ spreads out to $\mathbf{R^N}$ as $t \to \infty$.

(ii) The methods of the proof are same as those of Jimbo and Sakaguchi [14]. Assume $m = 1$ and let $u_0 \in \tilde{X}_\alpha \backslash \{0\}$. Then, for some $r_1 > 0$,

$$u_0(r) \text{ is a nonincreasing function in } r \geqslant r_1. \tag{3.46}$$

When $u_0(r_1) = 0$,

$$u_{0,\varepsilon}(r) = 0 \quad \text{for } \varepsilon > 0, r \geqslant r_1. \tag{3.47}$$

Hence, $u_{0,\varepsilon} \in X_\alpha$ and this case is reduced to case (i).

So, let $u_0(r_1) > 0$ and $\varepsilon_0 \in (0, u_0(r_1))$. Then, $u_{0,\varepsilon_0}(r_1) = u_0(r_1) - \varepsilon_0 = h > 0$. Hence, noting $u_0 \in L^\infty$ we have for small $t_1 > 0$,

$$\frac{h}{2} \leqslant u_\varepsilon(r_1, t) \leqslant \sup_{r \in [0,\infty)} u_\varepsilon(r, t) \leqslant 2||u_0||_\infty \quad \text{for } \varepsilon \in [0, \varepsilon_0], \ \ t \in [0, t_1]. \tag{3.48}$$

Therefore, by Suzuki [26], there exists $C > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$

$$u_\varepsilon(r, t) \leqslant Cr^{-\alpha} \quad \text{in } [0, \infty) \times [0, t_1]. \tag{3.49}$$

For $z \in \mathbf{R^N}$ and $v \in \mathbf{S^{N-1}}$ (i.e. $|v| = 1$), we put

$$A = A(z, v) = \{x \in \mathbf{R^N} \mid v \cdot (x - z) = 0\}, \tag{3.50}$$

where "$\cdot$" means the inner product in $\mathbf{R^N}$. A forms a hyperplane in $\mathbf{R^N}$. The upper (or lower) half-space of $\mathbf{R^N}$ with respect to A is defined as

$$\mathbf{R^N_{A,+}} = \{x \in \mathbf{R^N} \mid v \cdot (x - z) > 0\} [\text{or } \mathbf{R^N_{A,-}} = \{x \in \mathbf{R^N} \mid v \cdot (x - z) < 0\}]. \tag{3.51}$$

For any $x \notin A$, the reflection of x in A is denoted by $\sigma_A x$. Thus, we have for each $\zeta \in A$,

$$\zeta \cdot (\sigma_A x - x) = \tfrac{1}{2}(\sigma_A x + x) \cdot (\sigma_A x - x). \tag{3.52}$$

For any set $K \subset \mathbf{R^N}$, we define the reflection of K in A as

$$\sigma_A K = \{\sigma_A x \mid x \in K\} \tag{3.53}$$

and for any function v in $\mathbf{R^N}$, we define the reflection of v in A as

$$\sigma_A v(x) = v(\sigma_A x), \quad x \in \mathbf{R^N}. \tag{3.54}$$

Put $z(r)=(0,\dots,0,r)$, $\tilde{v}=(0,\dots,0,1)\in\mathbf{R^N}$ and $A(r)=A(z(r),\tilde{v})$. Then, because of $u_0(r_1)>0$ and (3.49), there exists $r_0\in(r_1,\infty)$ such that for any $r\geqslant r_0$ and $\varepsilon\in[0,\varepsilon_0]$

$$\sigma_{A(r)}u_\varepsilon(x,t)\leqslant u_\varepsilon(x,t) \quad \text{on } |x|=r_1,\ \ t\in[0,t_1]. \tag{3.55}$$

Moreover, noting $u_\varepsilon(x,t_1)>0$ in $\mathbf{R^N}$ we can choose above $r_0>0$ to satisfy

$$\sigma_{A(r)}u_\varepsilon(x,t_1)\leqslant u_\varepsilon(x,t_1) \quad \text{in } |x|<r_1. \tag{3.56}$$

By (3.46) we see that for any $r\geqslant r_0$ and $\varepsilon\in[0,\varepsilon_0]$,

$$\sigma_{A(r)}u_{0,\varepsilon}(x)\leqslant u_{0,\varepsilon}(x) \quad \text{in } \{|x|\geqslant r_1\}\cap\mathbf{R}^{\mathbf{N}}_{\mathbf{A(r)},-}. \tag{3.57}$$

Hence, since $v(x,t)=\sigma_{A(r)}u_\varepsilon(x,t)$ is also a solution of (1.1), the comparison theorem implies

$$\sigma_{A(r)}u_\varepsilon(x,t)\leqslant u_\varepsilon(x,t) \quad \text{in } r\in\{|x|\geqslant r_1\}\cap\mathbf{R}^{\mathbf{N}}_{\mathbf{A(r)},-},\ \ t\in[0,t_1]. \tag{3.58}$$

Therefore, by (3.56),

$$\sigma_{A(r)}u_\varepsilon(x,t_1)\leqslant u_\varepsilon(x,t_1) \quad \text{in } \mathbf{R}^{\mathbf{N}}_{\mathbf{A(r)},-}. \tag{3.59}$$

When $t_b(u_{0,\varepsilon})=\infty$, similarly, as in the proof of (3.58), we have for $r\geqslant r_0$,

$$\sigma_{A(r)}u_\varepsilon(x,t)\leqslant u_\varepsilon(x,t) \quad \text{in } \mathbf{R}^{\mathbf{N}}_{\mathbf{A(r)},-}\times[t_1,\infty). \tag{3.60}$$

When, $t_b(u_{0,\varepsilon})<\infty$, we can show (3.60) for approximate global solutions. Hence, by the limit procedure we get (3.60) for blow-up solutions.

Eq. (3.43) follows from (3.58) and (3.60). Note that $u_\varepsilon(x,t)$ is a radially symmetric function in $x\in\mathbf{R^N}$ and $t_c(u_{0,\varepsilon})=\infty$. Hence, by the similar methods to Lemma 4.3 of Suzuki [26] we get (3.42). Applying the maximum principle to $w=\sigma_{A(r)}u_\varepsilon(x,t)-u_\varepsilon(x,t)$, we obtain (3.44) by (3.58) and (3.60) (see [7,24]). Eq. (3.60) is also reduced to $u_\varepsilon(x,t)\in X_\alpha\subset\tilde{X}_\alpha$ for $0<t<t_b(u_{0,\varepsilon})$ and Lemmas 3.1 and 3.2 hold with X_α replaced by $\tilde{X}_\alpha$. The proof is complete. $\square$

4. Peaking solutions

In order to show the theorem we use peaking solutions which are constructed by Galaktionov and Vazquez [12]. A peaking solution is consist of a backward self-similar blow-up solution and a forward self-similar global solution of (1.1).

We first introduce backward self-similar blow-up solutions of (1.1). Let $T>0$ and

$$w_T(r,t)=(T-t)^{-\frac{1}{p-1}}\theta(\eta) \quad \text{with } \eta=\frac{r}{(T-t)^{\beta}}, \tag{4.1}$$

where $\beta = (p-m)/2(p-1) > 0$. If $w_T(r,t)$ $(r = |x|)$ is a solution of (1.1) and $\theta(\eta) > 0$ then $\theta(\eta)$ is a solution of the problem

$$\frac{1}{\eta^{N-1}}(\eta^{N-1}(\theta^m)')' - \beta\theta'\eta - \frac{\theta}{p-1} + \theta^p = 0 \quad (\eta > 0), \tag{4.2}$$

$$\theta'(0) = 0 \quad (\theta(0) = \mu > 0). \tag{4.3}$$

The existence of this $\theta(\eta)$ is guaranteed by the following lemma:

Lemma 4.1. *Let $m \geqslant 1$ and $p_s < p < p_p$. Then, there exits a solution $\theta(\eta)$ $(\eta > 0)$ such that*

$$\theta(\eta) = c_1\eta^{-\frac{2}{p-m}}(1 + o(1)) \quad \text{with } c_1 \in (0, c_s), \tag{4.4}$$

where

$$c_s = \left[\frac{2m}{p-m}\left(N - 2 - \frac{2m}{p-m}\right)\right]^{\frac{1}{p-m}}. \tag{4.5}$$

Hence, $w_T(r,t)$ is a backward self-similar solution which blows up at $t = T$.

Proof. See [12]. □

Forward self-similar global solutions of (1.1) are constructed as follows:

Lemma 4.2. *Let $m \geqslant 1$ and $p > p_{st} = mN/(N-2)$. Let $u_0(x) = cr^{-2/(p-m)}$ in $\mathbf{R^N}$ with $c \in (0, c_s)$. Then, there exists a solution $u(x,t)$ of* (1.1) *and* (1.2) *in $\mathbf{R^N} \times [0, \infty)$ such that*

$$u(x,t) = t^{-\frac{1}{p-1}} f_c(\zeta; \mu) \quad \left(\zeta = \frac{|x|}{t^\beta}\right), \tag{4.6}$$

where $f_c(0;\mu) = \mu$, $f'(0;\mu) = 0$ and $f(\eta;\mu) \in L^\infty([0,\infty))$. Further, if we put $f_c(\eta;\mu) = \mu V(y)$ and $y = \eta\mu^{\frac{p-m}{2}}$, then $V(y)$ satisfies

$$\begin{aligned} &\frac{1}{y^{N-1}}(y^{N-1}(V^m)')' + V^p \\ &\quad = -\mu^{1-p}\left(\beta V'y + \frac{V}{p-1}\right), \quad y > 0, \quad V(0) = 1, V'(0) = 0. \end{aligned} \tag{4.7}$$

Proof. See [12]. □

Thus, we can construct *peaking solutions* as in [12].

Proposition 4.3. *Put*

$$w_T(x,t) = (T-t)^{-\frac{1}{p-1}}\theta(\eta), \qquad \eta = \frac{|x|}{(T-t)^{\beta}} \quad \textit{in } (x,t)\in\mathbf{R}^{\mathbf{N}}\times(0,T),$$

$$w_T(x,T) = \begin{cases} c_1|x|^{-\frac{2}{p-m}} & \textit{in } x\neq 0, \\ \infty & \textit{on } x=0, \end{cases}$$

$$w_T(x,t) = (t-T)^{-\frac{1}{p-1}}f_{c_1}(\zeta;\mu), \qquad \zeta = \frac{|x|}{(t-T)^{\beta}} \quad \textit{in } (x,t)\in\mathbf{R}^{\mathbf{N}}\times(T,\infty),$$

where $c_1\in(0,c_s)$ and θ are as in Lemma 4.1 *and f_{c_1} is as in Lemma* 4.2. *Then $w_T(x,t)$ is a weak solution of* (1.1) *in $\mathbf{R}^{\mathbf{N}}\times(0,\infty)$. Further,*

$$w_T(x,t)\in C(\{\mathbf{R}^{\mathbf{N}}\times(0,\infty)\}\backslash\{(\mathbf{0},T)\}), \tag{4.8}$$

where $\mathbf{0}=(0,\dots,0)\in\mathbf{R}^{\mathbf{N}}$.

Proof. See [12]. $\square$

In the rest of this section, we study the intersection number between peaking solution $w_T(r,t)$ ($r=|x|$) and function

$$v_k(r) = \min\{h, kr^{-\alpha}\} \tag{4.9}$$

with $h>0$ and $0<k\leqslant k_0$. This is made in the following proposition and corollary which are used in the next section.

Let $\theta(r)\in C^1([0,\infty))$ satisfy the properties that

$$\theta(r) = cr^{-\gamma}(1+o(1)) \quad \text{as } r\to\infty, \tag{4.10}$$

where $c>0$, $\gamma>0$ and

$$\theta'(0)=0, \quad \theta(r)>0 \quad \text{in}[0,\infty). \tag{4.11}$$

Proposition 4.4. *Let $0<\gamma<\alpha$ and $0<k_0<\infty$. Put*

$$u_\tau(r) = \tau\theta(\tau^{1/\gamma}r) \quad (\tau>0). \tag{4.12}$$

Then, if τ is small enough, $v_k(r)$ $(0<k\leqslant k_0)$ and $u_\tau(r)$ intersect in $r\geqslant 0$ only at one point.

Proof. Since $0<\gamma<\alpha$, there exists $r_0>0$ such that for any $k\in(0,k_0]$

$$v_k(r)=kr^{-\alpha}\leqslant k_0r^{-\alpha}<\frac{c}{2}r^{-\gamma}\leqslant\theta(r)<h \quad \text{for } r\geqslant r_0. \tag{4.13}$$

Now, put

$$r_\tau=\tau^{-1/\gamma}r_0. \tag{4.14}$$

Since $\tau^{1/\gamma}r\geqslant r_0$ if and only if $r\geqslant r_\tau$, we see that for $\tau\in(0,1)$,

$$\begin{aligned} v_k(r)&<\frac{c}{2}r^{-\gamma}=\frac{c}{2}\tau(\tau^{1/\gamma}r)^{-\gamma}\\ &\leqslant u_\tau(r)=\tau\theta(\tau^{1/\gamma}r)<\tau h \quad \text{for } r\geqslant r_\tau(\geqslant r_0). \end{aligned} \tag{4.15}$$

That is,

$$v_k(r)<u_\tau(r)<h \quad \text{for } r\geqslant r_\tau(\geqslant r_0). \tag{4.16}$$

Further, if we choose $\tau_0\in(0,1)$ small, then for any $\tau\in(0,\tau_0]$

$$u_\tau(r)=\tau\theta(\tau^{1/\gamma}r)\leqslant\tau\sup_{0\leqslant\xi\leqslant r_0}\theta(\xi)<h=v_k(0) \quad \text{for } 0\leqslant r\leqslant r_\tau. \tag{4.17}$$

Hence, we see that $u_\tau(r)$ and $v(r)$ intersect in $r\geqslant 0$ at some points. Putting

$$m_0=\min_{0\leqslant r\leqslant r_0}\theta(r) \tag{4.18}$$

and choosing r_1 to satisfy $\tau m_0=kr_1^{-\alpha}$, namely,

$$r_1=\left(\frac{\tau m_0}{k}\right)^{-\frac{1}{\alpha}}. \tag{4.19}$$

We obtain the following two lemmas. $\square$

Lemma 4.5. *Let $0<\tau<\tau_0(<1)$. Let $\tilde r$ be the r-coordinate of an intersection point between $u_\tau(r)$ and $v_k(r)$ in $r\geqslant 0$. Then,*

$$\left(\frac{k}{h}\right)^{\frac{1}{\alpha}}<\tilde r\leqslant r_1. \tag{4.20}$$

Proof. We first note that

$$v_k(r)=\begin{cases} h & \text{if } 0\leqslant r\leqslant(\frac{k}{h})^{\frac{1}{\alpha}},\\ kr^{-\alpha} & \text{if } (\frac{k}{h})^{\frac{1}{\alpha}}<r. \end{cases} \tag{4.21}$$

Hence, from (4.16) and (4.17), we see that the left inequality of (4.20) holds.

Because of (4.16), we have

$$\tilde r<r_\tau. \tag{4.22}$$

Hence, if $r_1 \geqslant r_\tau$, then (4.20) is clear. If $r_1 < r_\tau$, then by the definitions of m_0 and r_1,

$$v_k(r) = kr^{-\alpha} < kr_1^{-\alpha} = \tau m_0 \leqslant \tau\theta(\tau^{1/\gamma}r) = u_\tau(r) < h \quad \text{for } r_1 < r \leqslant r_\tau, \tag{4.23}$$

and so by (4.16),

$$\tilde{r} \leqslant r_1. \tag{4.24}$$

The proof is complete. □

Lemma 4.6. *If τ is small enough, then for any $k \in (0, k_0]$,*

$$v_k'(r) < u_\tau'(r) \quad \text{in } [(k/h)^{\frac{1}{\alpha}}, r_1]. \tag{4.25}$$

Proof. First, by (4.20) we note

$$\left(\frac{k}{h}\right)^{\frac{1}{\alpha}} < r_1. \tag{4.26}$$

Put

$$M = \sup_{0 \leqslant r \leqslant r_0} |\theta'(r)|. \tag{4.27}$$

Then,

$$|u_\tau'(r)| = |\tau^{1+\frac{1}{\gamma}}\theta'(\tau^{\frac{1}{\gamma}}r)| \leqslant \tau^{1+\frac{1}{\gamma}}M \quad \text{for } 0 \leqslant r \leqslant r_\tau. \tag{4.28}$$

On the other hand, for $r \in [(k/h)^{1/\alpha}, r_1]$,

$$\begin{aligned} v_k'(r) = -\alpha k r^{-\alpha-1} \leqslant -\alpha k r_1^{-\alpha-1} &\leqslant -\alpha k\left(\frac{\tau m_0}{k}\right)^{\frac{\alpha+1}{\alpha}} \\ &= -\alpha k^{-1/\alpha} m_0^{(\alpha+1)/\alpha}\tau^{1+\frac{1}{\alpha}} \\ &\leqslant -\frac{\alpha k^{-1/\alpha} m_0^{(\alpha+1)/\alpha}}{M}\tau^{\frac{1}{\alpha}-\frac{1}{\gamma}}\tau^{1+\frac{1}{\gamma}}M. \end{aligned} \tag{4.29}$$

Since

$$\lim_{\tau \to 0} \tau^{\frac{1}{\alpha}-\frac{1}{\gamma}} = \infty \tag{4.30}$$

by inequality $\frac{1}{\alpha} - \frac{1}{\gamma} < 0$, we have, for small $\tau > 0$,

$$\frac{\alpha k^{-1/\alpha} m_0^{(\alpha+1)/\alpha}}{M}\tau^{\frac{1}{\alpha}-\frac{1}{\gamma}} \geqslant \frac{\alpha k_0^{-1/\alpha} m_0^{(\alpha+1)/\alpha}}{M}\tau^{\frac{1}{\alpha}-\frac{1}{\gamma}} > 1 \quad \text{for } 0 < k \leqslant k_0.$$

Hence,

$$v_k'(r) < -\tau^{1+\frac{1}{\gamma}} M \leqslant u_\tau'(r) \quad \text{for} \ \left(\frac{k}{h}\right)^{\frac{1}{\alpha}} \leqslant r \leqslant r_1. \tag{4.31}$$

The proof is complete. □

Proof of Proposition 4.4 (Continue). Let $\tau > 0$ be small enough. Then, Lemmas 4.5 and 4.6 hold. Let $\tilde{r}$ be the smallest r-coordinate of intersection points between $u_\tau(r)$ and $v_k(r)$ in $r \geqslant 0$. Then, by virtue of Lemmas 4.5 and 4.6, we have

$$v_k(r) < u_\tau(r) \quad \text{in } (\tilde{r}, r_1]. \tag{4.32}$$

Thus, we obtain the assertion of Proposition 4.4. □

Corollary 4.7. *Let* $2/(p-m) < \alpha$ *and* $k_0 > 0$. *Let* $w_T(r,t)$ $(r = |x|)$ *be a peaking solution in* $\mathbf{R}^\mathbf{N} \times (0, \infty)$, *which is constructed in Proposition* 4.3. *Then, if* T *is large enough,* $w_T(r,0)$ *and* $v_k(r)$ $(0 < k \leqslant k_0)$ *intersect in* $r \geqslant 0$ *only at one point and* $w_T(0,0) < v_k(0)$.

Proof. First, we note

$$w_T(r,0) = T^{-1/(p-1)} \theta(r T^{-(p-m)/2(p-1)}).$$

Put $\tau = T^{-1/(p-1)}$. Then,

$$w_T(r,0) = \tau\theta(r\tau^{1/\gamma})$$

where $\gamma = 2/(p-m)$. Since $\tau \downarrow 0$ if $T \to \infty$, by Proposition 4.4 we get the assertions of the corollary. The proof is complete. □

5. The case $u_0(x) = \min\{h, k|x|^{-\alpha}\}$

Let $h, k, \alpha > 0$ and put

$$v_{0,k}(r) = \min\{h, kr^{-\alpha}\}. \tag{5.1}$$

In this section, our aim is to construct an incomplete blow-up solution of (1.1) and (1.2) with $u_0(r) = v_{0,k}(r)$ for suitable $k > 0$ and $h > 0$. We shall show the next proposition. Then, Corollary 4.7 of Section 4 plays an important role.

Proposition 5.1. *Let* $2/(p-m) < \alpha < N$ *and* $p_s < p < p_p$. *Assume* $t_b(v_{0,k_1}) = \infty$ *for some* $k_1 > 0$ *and* $h > 0$. *Put* $v_k(x,t) = u(x,t;v_{0,k})$. *Then, there exits* $k_0 \in (k_1, \infty)$ *such that*

$$t_b(v_{0,k_0}) < \infty, \quad t_c(v_{0,k_0}) = \infty, \tag{5.2}$$

$$||v_{k_0}(\cdot,t)||_\infty = O(t^{-1/(p-1)}) \quad as \ t \to \infty. \tag{5.3}$$

Namely, $v_{k_0}(x,t)$ *is an incomplete blow-up solution of* (1.1).

Proof. Since $v_{0,k} \in X_\alpha$, by Lemma 3.1 we see that $v_k(\cdot, t) \in X_\alpha$ for $t \in (0, t_b(u_{0,k}))$. Put

$$k_0 = \sup\{k > 0 \mid t_b(v_{0,k}) = \infty\}. \tag{5.4}$$

Then, $0 < k_0 < \infty$. Furthermore, by Theorem 1 of [26] we have $t_c(v_{0,k}) = \infty$.

Next, we show (5.3). Let $w_T(x,t)$ be a peaking solution which is constructed in Section 4. By inequality $\alpha > 2/(p-m)$, Corollary 4.7 implies that if T is large enough, then for any $k \in (0, k_0)$, $w_T(r, 0)$ and $v_{0,k}(r)$ $(r = |x|)$ intersect in $r \geqslant 0$ only at one point and $w_T(0,0) < v_{0,k}(0)$.

Noting $\lim_{t \uparrow T} w_T(0,t) = \infty$, we see the existence of $t_1 = t_1(k) \in (0, T)$ satisfying

$$v_k(0, t_1) = w_T(0, t_1). \tag{5.5}$$

Hence, applying Proposition 3.3 with $\alpha_1 = 2/(p-m)$ and $\alpha_2 = \alpha$ to $w_T(x,t)$ and $v_k(x,t)$, we get

$$v_k(x,t) \leqslant w_T(x,t) \quad \text{in } \mathbf{R}^\mathbf{N} \times [t_1, \infty), \tag{5.6}$$

whence, for any $k \in (0, k_0)$,

$$v_k(x,t) \leqslant w_T(x,t) \quad \text{in } \mathbf{R}^\mathbf{N} \times [T, \infty). \tag{5.7}$$

Thus, noting $v_{k_0} = \lim_{k \uparrow k_0} v_k(x,t)$ by Proposition 2.9 we have

$$v_{k_0}(x,t) \leqslant w_T(x,t) \quad \text{in } \mathbf{R}^\mathbf{N} \times [T, \infty), \tag{5.8}$$

and, hence we get (5.3).

Finally, we show $t_b(v_{0,k_0}) < \infty$. Assume on the contrary that $t_b(v_{0,k_0}) = \infty$. Note that

$$v_{0,k} \downarrow v_{0,k_0} \quad \text{uniformly in } \mathbf{R}^\mathbf{N} \quad \text{as } k \downarrow k_0, \tag{5.9}$$

and for $k \in (0, k_0 + 1)$,

$$v_{0,k}(x) \leqslant \min\{h, (k_0+1)|x|^{-\alpha}\} \quad \text{in } \mathbf{R}^\mathbf{N}. \tag{5.10}$$

Then, as in the proof of Proposition 4.2 of Suzuki [25], there exist constants $\delta = \delta(T+1) > 0$ and $C = C(T+1) > 0$ such that for $k_0 < k \leqslant k_0 + \delta$

$$v_k(x,t) \leqslant C \quad \text{for } (x,t) \in \mathbf{R}^\mathbf{N} \times [0, T+1], \tag{5.11}$$

$$v_k(x,t) \downarrow v_{k_0}(x,t) \quad \text{locally uniformly in } \mathbf{R}^\mathbf{N} \times [0, T+1] \text{ as } k \downarrow k_0. \tag{5.12}$$

Hence, because of Lemma 3.2, for some $C' > 0$

$$v_{k_0+\delta}(x,t)|x|^\alpha \leqslant C' \quad \text{in } \mathbf{R}^\mathbf{N} \times [0, T+1]. \tag{5.13}$$

On the other hand, because of Lemma 3.2 again, for some $c > 0$

$$w_T(x, T+1) \geqslant c|x|^{-\frac{2}{p-m}} \quad \text{in } |x| \geqslant 1. \tag{5.14}$$

Since $v_{k_0}(x, T) \not\equiv w_T(x, T)$, (5.8) and the maximum principle imply

$$v_{k_0}(x, t) < w_T(x, t) \quad \text{in } \mathbf{R}^{\mathbf{N}} \times (T, \infty). \tag{5.15}$$

Thus, noting $2/(p-m) < \alpha$, by (5.13) and (5.14) we see that there exists $r_0 > 0$ such that for any $k \in (k_0, k_0 + \delta]$,

$$v_k(x, T+1) \leqslant v_{k_0+\delta}(x, T+1) \leqslant w_T(x, T+1) \quad \text{in } |x| \geqslant r_0. \tag{5.16}$$

Therefore, by (5.12) and (5.15), there exists $k_2 \in (k_0, k_0 + \delta)$ such that for $k_0 < k < k_2$,

$$v_k(x, T+1) \leqslant w_T(x, T+1) \quad \text{in } \mathbf{R}^{\mathbf{N}}, \tag{5.17}$$

whence by the comparison theorem,

$$v_k(x, t) \leqslant w_T(x, t) < \infty \quad \text{in } \mathbf{R}^{\mathbf{N}} \times [T+1, \infty). \tag{5.18}$$

Hence, $t_b(v_{0,k}) = \infty$ for $k_0 < k < k_2$. This is a contradiction to the definition of k_0 and we get $t_b(v_{0,k_0}) < \infty$. The proof is complete. $\square$

6. Proof of Theorem 2.5

In this section, we prove Theorem 2.5 in a series of propositions. The methods of the proof are the same as those of Proposition 5.1. Set

$$D_\alpha = \{u_0 \in X_\alpha; t_b(u_0) = \infty\}. \tag{6.1}$$

Proposition 6.1. *Let $p_s < p < p_p$ and $2/(p-m) < \alpha < N$. Then, D_α is an open set in X_α and $u_0 \equiv 0 \in \operatorname{Int}(D_\alpha) = D_\alpha$, where $\operatorname{Int}(D_\alpha)$ is the interior of D_α in X_α.*

Proof. By inequality $2/(p-m) < \alpha$, we note that for some $C > 0$, $||v||_{N(p-m)/2} \leqslant C||v||_{\infty,\alpha}$. Hence, it follows from Theorem 4.1 of Kawanago [17] that if $||u_0||_{\infty,\alpha}$ is small enough then $t_b(u_0) = \infty$. This fact shows that $u_0 \equiv 0$ is in $\operatorname{Int}(D_\alpha)$.

Let $u_0 \in D_\alpha \backslash \{0\}$ and put $u(x, t) = u(x, t; u_0)$. Then, by Proposition 3.4 we see that (3.42)–(3.45) hold with $\varepsilon = 0$ for some $r_0 > 0, t_0 > 0$ and $\delta > 0$. Hence, for $h \in (0, \delta)$, $u = h$ and $u = u(r, t_0) = u(x, t_0)$ $(r = |x|)$ intersect in $r \geqslant 0$ only at one point. Further, by virtue of Lemma 3.2, we get for some $C > 0$,

$$u(r, t_0) \leqslant Cr^{-\alpha} \quad \text{for } r \geqslant r_0, \tag{6.2}$$

whence for $\alpha' \in (2/(p-m), \alpha)$ there exists $k_1 > 0$ such that

$$u(r, t_0) \leqslant k_1 r^{-\alpha'} \quad \text{for } r \geqslant r_0. \tag{6.3}$$

Thus, putting for $0 < h < \delta$,

$$v_{0,k}(r) = \min\{h, kr^{-\alpha'}\} \tag{6.4}$$

we see that $v_{0,k}(r)$ $(k \geqslant k_1)$ and $u(r, t_0)$ intersect in $r \geqslant 0$ only at one point. Further, by Theorem 4.1 of Kawanago [17], there exists $h_0 \in (0, \delta)$ such that if $h = h_0$ then $t_b(v_{0,k_1}) = \infty$ (see above).

Let $h = h_0$ and put $v_k(x,t) = u(x,t; v_{0,k})$. Then, by Proposition 5.1, there exists $k_0 \in (k_1, \infty)$ such that $t_b(v_{0,k_0}) < \infty$ and

$$||v_{k_0}(\cdot, t)||_\infty = O(t^{-1/(p-1)}) \quad \text{as } t \to \infty. \tag{6.5}$$

Hence, for some $t_1 \in (0, t_b(v_{0,k_0}))$,

$$v_{k_0}(0, t_1) = u(0, t_1 + t_0). \tag{6.6}$$

From Proposition 3.3 we have

$$u(r,t) \leqslant v_{k_0}(r, t - t_0) \quad \text{for } r \geqslant 0, t \geqslant t_1 + t_0. \tag{6.7}$$

Therefore, by (6.5), there exists $t_2 > t_b(v_{0,k_0})$ such that

$$u(x, t_2 + t_0) \leqslant v_{k_0}(x, t_2) < \infty \quad \text{in } \mathbf{R}^\mathbf{N}, \tag{6.8}$$

whence, the maximum principle implies

$$u(x, t_2 + t_0) < v_{k_0}(x, t_2) < \infty \quad \text{in } \mathbf{R}^\mathbf{N}. \tag{6.9}$$

On the other hand, by Lemma 3.2 we note that for some $C > 1$,

$$u(r, t_2 + t_0) \leqslant Cr^{-\alpha} \quad \text{for } r \geqslant r_0, \tag{6.10}$$

$$v_{k_0}(r, t_2) \geqslant C^{-1} r^{-\alpha'} \quad \text{for } r \geqslant r_0. \tag{6.11}$$

Now, put $\varepsilon > 0$,

$$u_{0,\varepsilon}(x) = u_0(x) + \varepsilon(1 + |x|)^{-\alpha}. \tag{6.12}$$

Then, clearly $u_{0,\varepsilon} \in X_\alpha$ and

$$u_{0,\varepsilon}(x) \downarrow u_0(x) \quad \text{uniformly in } \mathbf{R}^\mathbf{N} \text{ as } \varepsilon \downarrow 0. \tag{6.13}$$

Further, put $u_\varepsilon(x,t) = u_\varepsilon(x,t;u_{0,\varepsilon})$. Then, as in the proof of Proposition 5.1, we see

$$u_\varepsilon(x,t)\downarrow u(x,t) \quad \text{uniformly in } \mathbf{R}^\mathbf{N}\times[0,t_2+t_0] \text{ as } \varepsilon\downarrow 0, \tag{6.14}$$

and if ε is small enough then

$$u_\varepsilon(x,t)\leqslant v_{k_0}(x,t-t_0) \quad \text{in } \mathbf{R}^\mathbf{N}\times[t_2+t_0,\infty), \tag{6.15}$$

whence $t_b(u_{0,\varepsilon}) = \infty$. Therefore, if $||u_0-\tilde{u}_0||_{\infty,\alpha}<\varepsilon$ and $\tilde{u}_0\in X_\alpha$, then

$$\tilde{u}_0(x)\leqslant(1+|x|)^{-\alpha}\varepsilon+u_0(x) = u_{0,\varepsilon}(x) \quad \text{in } \mathbf{R}^\mathbf{N}, \tag{6.16}$$

and so $t_b(\tilde{u}_0) = \infty$. Namely, $\tilde{u}_0\in D_\alpha$, and hence $u_0\in \mathrm{Int}(D_\alpha)$.

Thus, we see that D_α is an open set in X_α. $\square$

Proposition 6.2. *Assume $p_s<p<p_p$ and $2/(p-m)<\alpha<N$. Let $u_0\in X_\alpha$ and put $u(x,t) = u(x,t;u_0)$. If $t_c(u_0) = \infty$, then*

$$||u(\cdot,t)||_\infty = O(t^{-1/(p-1)}) \quad as\ t\to\infty. \tag{6.17}$$

Proof. When $u_0\equiv 0$, (6.17) is obvious.

Let $u_0\not\equiv 0$. Put for $\varepsilon>0$,

$$u_{0,\varepsilon}(x) = [u_0(x)-\varepsilon]_+ \tag{6.18}$$

and $u_\varepsilon(x,t) = u(x,t;u_{0,\varepsilon})$. Then, we see

$$t_b(u_{0,\varepsilon}) = \infty. \tag{6.19}$$

In fact, assume on the contrary that $t_b(u_{0,\varepsilon})<\infty$. Since $u_{0,\varepsilon}\in C_0(\mathbf{R}^\mathbf{N})$, there exists $\delta>0$ such that if $|x'|<\delta$ then

$$u_{0,\varepsilon}(x+x')\leqslant u_0(x) \quad \text{in } \mathbf{R}^\mathbf{N}, \tag{6.20}$$

(see the proof of Proposition 3.8 of [26]) from which

$$u_\varepsilon(x+x',t)\leqslant u(x,t) \quad \text{in } \mathbf{R}^\mathbf{N}\times(0,t_b(u_{0,\varepsilon})). \tag{6.21}$$

Therefore, as in the proof of Proposition 3.8 of [26], we have $t_c(u_0)\leqslant t_b(u_{0,\varepsilon})<\infty$. This is a contradiction to the assumption and so $t_b(u_{0,\varepsilon}) = \infty$.

By Proposition 3.4 we have (3.42)–(3.45) for some $r_0>0, \varepsilon_0>0, t_0>0$ and $\delta>0$. Let $\alpha'\in(2/(p-m),\alpha)$. Then, by virtue of Lemma 3.2, there exists $k_1>0$ such that for any $\varepsilon > 0$,

$$u_\varepsilon(r, t_0) \leqslant u(r, t_0) \leqslant k_1 r^{-\alpha'} \quad \text{for } r \geqslant r_0. \tag{6.22}$$

Putting $v_{0,k}(r) = \min\{h, kr^{-\alpha'}\}$ $(0 < h < \delta)$ we see that for any $\varepsilon \in (0, \varepsilon_0)$, $v_{0,k}(r)$ $(k \geqslant k_1)$ and $u_\varepsilon(r, t_0)$ intersect in $r \geqslant 0$ only at one point and $u_\varepsilon(0, t_0) > v_{0,k}(0)$. Let $h \in (0, \delta)$ satisfy $t_b(v_{0,k_1}) = \infty$ and put $v_k(r, t) = u(x, t; v_{0,k})$ $(r = |x|)$. Then, as in the proof of Proposition 6.1, we get $t_b(v_{0,k_0}) < \infty$ and (6.5) for some $k_0 \in (k_1, \infty)$. Further, as in the proof of Proposition 6.1, there exists $t_2 > t_b(v_{0,k_0})$ such that for any $\varepsilon > 0$

$$u_\varepsilon(r, t) \leqslant v_{k_0}(r, t - t_0) \leqslant C(t - t_0)^{-\frac{1}{p-1}} \quad \text{for } r \geqslant 0, \ \ t \geqslant t_2 + t_0. \tag{6.23}$$

Thus, if $\varepsilon \downarrow 0$, then

$$u(r, t) \leqslant v_{k_0}(r, t - t_0) \leqslant C(t - t_0)^{-\frac{1}{p-1}} \quad \text{for } r \geqslant 0, \ \ t \geqslant t_2 + t_0. \qquad \square \tag{6.24}$$

Proposition 6.3. *Assume $p_s < p < p_p$ and $2/(p - m) < \alpha < N$. Then,*

$$\partial D_\alpha = \{u_0 \in X_\alpha; t_b(u_0) < \infty, t_c(u_0) = \infty\}, \tag{6.25}$$

where ∂D_α is the boundary of D_α in X_α.

Proof. We first show

$$\partial D_\alpha \subset \{u_0 \in X_\alpha; t_b(u_0) < \infty, t_c(u_0) = \infty\}. \tag{6.26}$$

Let $u_0 \in \partial D_\alpha$. Then, since D_α is an open set in X_α (Proposition 6.1), $t_b(u_0) < \infty$. By the definition of ∂D_α, there exists a sequence of functions $\{u_{0,n}\} \subset D_\alpha$ such that

$$u_{0,n} \to u_0 \quad \text{in } L^\infty_\alpha \ \text{ as } n \to \infty. \tag{6.27}$$

Putting

$$\tilde{u}_{0,n}(x) = \inf_{n' \geqslant n} u_{0,n'}(x) \quad \text{for } x \in \mathbf{R}^\mathbf{N}, \tag{6.28}$$

we see that $\tilde{u}_{0,n}(x) \leqslant u_{0,n}(x)$ in $\mathbf{R}^\mathbf{N}$, $t_b(\tilde{u}_{0,n}) = \infty$ and

$$\tilde{u}_{0,n}(x) \uparrow u_0(x) \quad \text{as } n \to \infty \ \text{ for each } x \in \mathbf{R}^\mathbf{N}. \tag{6.29}$$

Hence, putting $\tilde{u}_n(x, t) = u(x, t; \tilde{u}_{0,n})$ and $u(x, t) = u(x, t; u_0)$, by Proposition 2.9 we get

$$\tilde{u}_n(x, t) \uparrow u(x, t) \quad \text{as } n \to \infty \ \text{ for each } (x, t) \in \mathbf{R}^\mathbf{N} \times (0, \infty). \tag{6.30}$$

On the other hand, it follows from (iii) of Proposition 2.4 that for some $\mu_R > 0$,

$$\int_{B_R} \tilde{u}_n(x,t)\psi_R(x)\,dx \leqslant \mu_R \quad \text{for } t>0, n\geqslant 1, \tag{6.31}$$

where $B_R = \{|x|<R\}$ $(R>0)$ and ψ_R is the first eigenfunction of $-\Delta$ in B_R with Dirichlet boundary condition. Therefore, if $n\to\infty$, then

$$\int_{B_R} u(x,t)\psi_R(x)\,dx \leqslant \mu_R \quad \text{for } t>0, \tag{6.32}$$

whence $t_c(u_0) = \infty$. Thus, we get (6.26).

Next, we shall show

$$\partial D_\alpha \supset \{u_0 \in X_\alpha; t_b(u_0) < \infty, t_c(u_0) = \infty\}. \tag{6.33}$$

Let $u_0 \in X_\alpha$, $t_b(u_0) < \infty$ and $t_c(u_0) = \infty$. Then, by Proposition 3.4 we get (3.42)–(3.44) with $\varepsilon = 0$ for some $r_0 > 0$.

We first consider the case where $u_0(x)$ is of compact support in $\mathbf{R}^\mathbf{N}$: $u_0(x) \in C_0(\mathbf{R}^\mathbf{N})$. In this case, putting for $\varepsilon > 0$, $u_{0,\varepsilon}(x) = [u_0(x) - \varepsilon]_+$, we see

$$u_{0,\varepsilon} \to u_0 \quad \text{in } L^\infty_\alpha \text{ as } \varepsilon \downarrow 0. \tag{6.34}$$

Therefore, since $t_b(u_{0,\varepsilon}) = \infty$ by the proof of Proposition 6.2, we get $u_{0,\varepsilon} \in D_\alpha$ and hence $u_0 \in \partial D_\alpha$.

Next, we consider the case where $u_0(x) \notin C_0(\mathbf{R}^\mathbf{N})$. Then, $u_0(r) > 0$ in $r \geqslant 0$. Let $0 < \varepsilon < \min_{0\leqslant r\leqslant r_0} u_0(r)$. Then, there exists $r_1 = r_1(\varepsilon) > r_0$ such that

$$u_0(r) > u_0(r_0) - \varepsilon \quad \text{in } r_0 < r < r_1 \quad \text{and} \quad u_0(r_1) = u_0(r_0) - \varepsilon. \tag{6.35}$$

Put

$$u_{0,\varepsilon}(r) = \begin{cases} u_0(r) - \varepsilon & \text{in } 0 \leqslant r \leqslant r_0, \\ u_0(r_0) - \varepsilon & \text{in } r_0 < r \leqslant r_1, \\ u_0(r) & \text{in } r > r_1. \end{cases} \tag{6.36}$$

Then, clearly $u_{0,\varepsilon} \in X_\alpha$. Further, similarly, as in the proof of Proposition 4.2 of [26], we can show $t_b(u_{0,\varepsilon}) = \infty$ (see also the proof of Proposition 6.4).

Thus, noting that

$$u_{0,\varepsilon} \to u_0 \quad \text{in } L^\infty_\alpha \text{ as } \varepsilon \downarrow 0, \tag{6.37}$$

we obtain $u_0 \in \partial D_\alpha$. Therefore, we have (6.33). The proof is complete. □

Proposition 6.4. *When, $m = 1$, Propositions 6.1–6.3 hold with X_α replaced by $\tilde{X}_\alpha$.*

Proof. Let $m = 1$. Noting Proposition 3.2(ii), by the same methods as those of the proofs when $u_0 \in X_\alpha$, we can show that Propositions 6.1 and 6.2 hold with X_α replaced by $\tilde{X}_\alpha$.

So, we show only Proposition 6.3 with X_α replaced by $\tilde{X}_\alpha$. As in the proof of Proposition 6.3, it is not difficult to see that

$$\partial\tilde{D}_\alpha \subset \{u_0 \in \tilde{X}_\alpha;\, t_b(u_0) < \infty,\, t_c(u_0) = \infty\}, \tag{6.38}$$

where

$$\tilde{D}_\alpha = \{u_0 \in \tilde{X}_\alpha;\, t_b(u_0) = \infty\} \tag{6.39}$$

and $\partial\tilde{D}_\alpha$ is the boundary of $\tilde{D}_\alpha$ in $\tilde{X}_\alpha$.

We shall show

$$\partial\tilde{D}_\alpha \supset \{u_0 \in \tilde{X}_\alpha;\, t_b(u_0) < \infty,\, t_c(u_0) = \infty\}. \tag{6.40}$$

The methods of the proof are similar to those of the proof of Proposition 6.3. Let $u_0 \in \tilde{X}_\alpha$, $t_b(u_0) < \infty$ and $t_c(u_0) = \infty$. Then, by Proposition 3.4(ii) we get (3.42)–(3.44) with $\varepsilon = 0$ for some $r_0 > 0$.

We first consider the case where $u_0(r_0) = 0$. In this case, putting for $\varepsilon > 0$, $u_{0,\varepsilon}(x) = [u_0(x) - \varepsilon]_+$, we see

$$u_{0,\varepsilon} \to u_0 \quad \text{in } L^\infty_\alpha \text{ as } \varepsilon \downarrow 0. \tag{6.41}$$

Therefore, since $t_b(u_{0,\varepsilon}) = \infty$ by the proof of Proposition 6.2, we get $u_{0,\varepsilon} \in \tilde{D}_\alpha$ and hence $u_0 \in \partial\tilde{D}_\alpha$.

Next, we consider the case where $u_0(r_0) > 0$. Let $0 < \varepsilon < u_0(r_0)$. Then, there exists $r_1 = r_1(\varepsilon) > r_0$ such that

$$u_0(r) > u_0(r_0) - \varepsilon \quad \text{in } r_0 < r < r_1 \quad \text{and} \quad u_0(r_1) = u_0(r_0) - \varepsilon. \tag{6.42}$$

Put

$$u_{0,\varepsilon}(r) = \begin{cases} [u_0(r) - \varepsilon]_+ & \text{in } 0 \leqslant r \leqslant r_0, \\ u_0(r_0) - \varepsilon & \text{in } r_0 < r \leqslant r_1, \\ u_0(r) & \text{in } r > r_1. \end{cases} \tag{6.43}$$

Then, clearly $u_{0,\varepsilon} \in \tilde{X}_\alpha$. Further, we can show $t_b(u_{0,\varepsilon}) = \infty$. In fact, assume on the contrary that $t_b(u_{0,\varepsilon}) < \infty$. Put $u_\varepsilon(x,t) = u(x,t;u_{0,\varepsilon})$ and $u(x,t) = u(x,t;u_0)$. Then, since

$$u_{0,\varepsilon}(r) \leqslant u_0(r) \quad \text{in } r \geqslant 0, \tag{6.44}$$

the comparison theorem implies

$$u_\varepsilon(r,t) \leqslant u(r,t) \quad \text{in } r \geqslant 0, t > 0. \tag{6.45}$$

Hence, because of (3.42) with $\varepsilon = 0$, the maximum principle implies

$$u_\varepsilon(x,t) < u(x,t) \quad \text{for } r_0 < |x| < r_1, t \geqslant 0. \tag{6.46}$$

Now, we put

$$v_{0,\varepsilon}(r) = \begin{cases} u_0(r) - \varepsilon & \text{in } 0 \leqslant r \leqslant r_0, \\ u_0(r_0) - \varepsilon & \text{in } r_0 < r \leqslant r_1, \\ u_0(r) & \text{in } r > r_1. \end{cases} \tag{6.47}$$

Then, $[v_{0,\varepsilon}]_+ = u_{0,\varepsilon}$ and

$$v_{0,\varepsilon}(x) < u_0(x) \quad \text{in } 0 \leqslant |x| < r_1. \tag{6.48}$$

We further put for $x \in \mathbf{R}^\mathbf{N}$,

$$\tilde{u}_\varepsilon(x,t) = \begin{cases} v_{0,\varepsilon}(x) & \text{if } t = 0, \\ u_\varepsilon(x,t) & \text{if } t > 0. \end{cases} \tag{6.49}$$

Then, $u(x,t)$ and $\tilde{u}_\varepsilon(x,t)$ are continuous in $\mathbf{R}^\mathbf{N} \times \{0\} \cup \{|x| > r_0\} \times \{t \geqslant 0\}$. Hence, letting $\tilde{r} = (r_0 + r_1)/2$ and $0 < \eta < (r_1 - r_0)/2$ and putting

$$K = \{(x,t,u(x,t)) \mid (x,t) \in \{|x| \leqslant \tilde{r} + \eta\} \times \{0\}$$
$$\cup \{\tilde{r} - \eta \leqslant |x| \leqslant \tilde{r} + \eta\} \times \{0 \leqslant t \leqslant t_b(u_{0,\varepsilon})\}\}$$

and

$$L = \{(x,t,\tilde{u}_\varepsilon(x,t)) \mid (x,t) \in \{|x| \leqslant \tilde{r}\} \times \{0\} \cup \{|x| = \tilde{r}\} \times \{0 \leqslant t \leqslant t_b(u_{0,\varepsilon})\}\},$$

we see from (6.46) and (6.48), that K and L are bounded closed set in $\mathbf{R}^\mathbf{N}$, and $K \cap L = \emptyset$. Hence, $\operatorname{dist}(K,L) > 0$, where $\operatorname{dist}(K,L)$ is the distance between K and L

in $\mathbf{R}^{\mathbf{N+2}}$. Therefore, let $0<\delta<\min\{\mathrm{dist}(K,L)/2,\eta\}$. Then, if $|x'|<\delta$, we have

$$\{|x|\leqslant\tilde{r}\}+\{x'\}\equiv\{x+x'\mid|x|\leqslant\tilde{r}\}\subset\{|x|\leqslant\tilde{r}+\eta\}, \tag{6.50}$$

$$\{|x|=\tilde{r}\}+\{x'\}\subset\{\tilde{r}-\eta\leqslant|x|\leqslant\tilde{r}+\eta\}, \tag{6.51}$$

$$\begin{aligned}&\tilde{u}_\varepsilon(x,t)<u(x+x',t)\\&\quad\text{in }(x,t)\in\{|x|\leqslant\tilde{r}\}\times\{0\}\cup\{|x|=\tilde{r}\}\times[0,t_b(u_{0,\varepsilon})],\end{aligned} \tag{6.52}$$

that is,

$$\begin{aligned}&u_\varepsilon(x,t)=[\tilde{u}_\varepsilon(x,t)]_+\leqslant u(x+x',t)\\&\quad\text{in }(x,t)\in\{|x|\leqslant\tilde{r}\}\times\{0\}\cup\{|x|=\tilde{r}\}\times[0,t_b(u_{0,\varepsilon})].\end{aligned} \tag{6.53}$$

Hence, the comparison theorem implies that if $|x'|<\delta$ then

$$u_\varepsilon(x,t)\leqslant u(x+x',t)\quad\text{in }(x,t)\in\{|x|\leqslant\tilde{r}\}\times[0,t_b(u_{0,\varepsilon})]. \tag{6.54}$$

Here, we note

$$\lim_{t\uparrow t_b(u_{0,\varepsilon})}||u_\varepsilon(\cdot,t)||_{L^\infty(|x|\leqslant\tilde{r})}=\infty. \tag{6.55}$$

Therefore, as in the proof of Proposition 3.8 of [26], we see

$$t_c(u_0)\leqslant t_b(u_{0,\varepsilon})<\infty. \tag{6.56}$$

This is a contradiction to $t_c(u_0)=\infty$, and hence $t_b(u_{0,\varepsilon})=\infty$, namely $u_{0,\varepsilon}\in\tilde{D}_\alpha$. Thus, noting that

$$u_{0,\varepsilon}\to u_0\quad\text{in }L^\infty_\alpha\text{ as }\varepsilon\downarrow 0, \tag{6.57}$$

we obtain $u_0\in\partial\tilde{D}_\alpha$. Therefore, we have (6.40). The proof is complete. □

Proposition 6.5. *Assume* $p_s<p<p_p$ *and* $2/(p-m)<\alpha<N$. *Let* $u_0\in\tilde{X}_\alpha\backslash\{0\}$ *when* $m=1$ *and* $u_0\in X_\alpha^+$ *when* $m>1$. *Let* $v_0\in C(\mathbf{R}^\mathbf{N})$. *Then we obtain the following*:

(i) *Suppose* $t_b(u_0) < \infty$. *If* $v_0 \geqslant u_0$ *and* $v_0 \not\equiv u_0$ *in* $\mathbf{R}^\mathbf{N}$ (*and further* $v_0 \not\equiv u_0$ *in* $\operatorname{supp} u_0$ *when* $m > 1$), *then* $t_c(v_0) < \infty$.

(ii) *Suppose* $t_c(u_0) = \infty$. *If* $v_0 \leqslant u_0$ *and* $v_0 \not\equiv u_0$ *in* $\mathbf{R}^\mathbf{N}$ (*and further* $v_0 \not\equiv u_0$ *in* $\operatorname{supp} u_0$ *when* $m > 1$), *then* $t_b(v_0) = \infty$.

(iii) *Hence*, (2.15) *of Theorem* 2.5 *holds, and when* $m = 1$ (2.15) *holds with* X_α *and* X_α^+ *replaced by* $\tilde{X}_\alpha$ *and* $\tilde{X}_\alpha \backslash \{0\}$, *respectively*.

Proof. (i) We first consider the case $m > 1$. Let $u_0 \in X_\alpha^+$, $v_0 \geqslant u_0$ in $\mathbf{R}^\mathbf{N}$ and $v_0 \not\equiv u_0$ in $\operatorname{supp} u_0$. Then, if we interchange conditions on u_0 and v_0, it is not difficult to see that u_0 and v_0 satisfy the assumptions of Theorem 2.7 of [26] for some $D = B_R$ $(R > r_0)$. Hence, by Theorem 2.7 of [26], we get $t_c(v_0) \leqslant t_b(u_0)$. Therefore, if $t_b(u_0) < \infty$ then $t_c(v_0) < \infty$.

Next, we consider the case $m = 1$. Let $u_0 \in \tilde{X}_\alpha \backslash \{0\}$ and $u(x,t) = u(x,t;u_0)$. Then, we note that $u(x,t) > 0$ in $\mathbf{R}^\mathbf{N} \times (0,\infty)$. Hence, by the proof of (ii) of Proposition 3.4 we get $u(\cdot,t) \in X_\alpha^+$ for $t \in (0, t_b(u_0))$. Put $v(x,t) = u(x,t;v_0)$. Then, if we consider $u(x,t_1)$ and $v(x,t_1)$ $(0 < t_1 < \min\{t_b(u_0), t_b(v_0)\})$ as initial data $u_0(x)$ and $v_0(x)$, respectively, by the positivity of solutions we can get the assertion of (i) as above.

(ii) The methods of the proof are the same as those of (i) and we omit the proof.

(iii) We first consider the case $m > 1$. We note, by Propositions 6.1 and 6.3, that $\operatorname{Int}(K_\alpha) = D_\alpha$ and $\partial K_\alpha = \partial D_\alpha$. Let $u_0 \in X_\alpha^+$. Then, by $u_0 \equiv 0 \in D_\alpha$ and the blow-up theorem, it is not difficult to see that $\tau u_0 \in D_\alpha$ if τ is small enough and $\tau u_0 \notin K_\alpha$ if τ is large enough. Hence, there exists $\tau_0 > 0$ such that $\tau_0 u_0 \in \partial K_\alpha = \partial D_\alpha$, that is, $t_b(\tau_0 u_0) < \infty$ and $t_c(\tau_0 u_0) = \infty$. Thus, by virtue of (i) and (ii) above, if $0 < \tau < \tau_0$ then $\tau u_0 \in D_\alpha = \operatorname{Int}(K_\alpha)$, and if $\tau > \tau_0$ then $\tau u_0 \in C_\alpha = X_\alpha \backslash K_\alpha$. The case $m = 1$ is also proved by the same methods. The proof is complete. $\square$

Proof of Theorem 2.5. (i) By Proposition 6.3, we see $K_\alpha = D_\alpha \cup \partial D_\alpha$. Hence, it is clear that K_α is a closed subset in X_α, C_α is an open subset in X_α and $0 \in \operatorname{Int}(K_\alpha)$. By the proof of Theorem 1 of Kawanago [16], it is also clear that C_α, $\operatorname{Int}(K_\alpha)$ and K_α are unbounded subsets in X_α.

Property (ii) follows from Lemma 3.1.

(iii) Eq. (2.15) follows from (iii) of Proposition 6.5. Similarly, as in the proof of Theorem 1 of Kawanago [17], we can prove the rest of the assertions.

Property (iv) follows from Proposition 6.2.

Property (v) follows from Propositions 6.1 and 6.3. $\square$

Proof of Corollary 2.6. By Propositions 6.4 and 6.5, the methods of the proof are the same as those of the proof of Theorem 2.5 and we omit the proof. $\square$

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